Patterning and Algebra Breakout Sessions

There are three key ideas that we will focus on during the course of the two-day breakout sessions: multiplicative reasoning, generalizing, and multiple representations (prioritizing visual in order to understand and *see* mathematical structure). We will also infuse examples of how to ask questions of students that allows them to formulate conjectures about mathematical relationships and mathematical structure.

We have chosen these three key interconnected concepts because these are the areas that are fundamentally important, but with which students have the most difficult in terms of their algebraic learning.

Multiplicative Thinking:

Research indicates that while most students are able to solve multiplication problems involving small numbers, they tend to rely on additive strategies to solve more complex problems (Siemon & Breed). The transition from additive to multiplicative thinking is neither smooth nor straightforward. When considering the development of algebraic reasoning, multiplicative thinking is fundamental to understanding the idea of "function" as the co-variation between two sets of data – also known as one-to-many correspondence. Many traditional approaches and models for algebraic instruction seem to (inadvertently) support additive thinking, that is, a recognition of variation within one set of data (for instance, *for this pattern add three more tiles each time*). Multiplicative thinking, on the other hand, is the recognition of co-variation between two sets of data (for instance, *for any term/position number of the pattern multiply the term by 3 to figure out the number of tiles*).

Generalizing:

Algebra can be thought of as generalizations of laws about relationships between and among numbers and patterns. Part of this understanding includes working with letters as expressions of variables in order to represent *any* case in a relationship – not just a particular missing unknown quantity. A focus is on transitioning from term-to-term generalizations (scalar/recursive or additive thinking) to generalizations that result in explicit statements about mathematical relations between independent and dependent variables. The ability to generalize includes an ability to make near predictions, for instance the 5^{th} term of a pattern, far predictions, for instance the 28^{th} term of a pattern. It also includes the ability to be able to articulate the general rule for a pattern in order to be able to make predictions for *any* term.

Multiple Representations

Observation of patterns and relationships lie at the heart of acquiring deep understanding of algebraic reasoning. Past research indicates that working with visual representations (linear growing patterns, diagrams) and deconstructing these in order to identify the relationship between variables have been more successful methods of developing generalized algebraic formulae than either working with number sequences, using ordered tables of values, or memorizing rules for transforming equations. In addition, research suggests that when data is expressed numerically and sequentially (ordered table of values) scalar or term-to-term relationships are likely to be dominant.

When considering the study of algebraic relationships in higher grades, mathematics educators recommend that students be introduced to various representational forms of relationships in order to develop the ability to use these representations effectively as a means of considering quantitative relationships (e.g., Janvier, 1987a, 1987b; Moschkovich, Schoenfeld & Arcavi, 1993), including symbolic notation and graphs. In addition, researchers stress that it is the ability to make *connections* among different representations, specifically symbolic/numeric and graphic ones, that allow students to develop insights for constructing the concept of a mathematical relationship (e.g., Evan, 1998; Bloch, 2003).

For higher grades, we will also be exploring how to support students in constructing an understanding of solving [systems of] equations of the form ax+b=cx+d for which there are one solution, no solution, or infinite solutions. This exploration will be based on their understanding of linear growing patterns and graphs of linear growing patterns. Research has demonstrated that students who rely only on remembered rules for transforming and solving equations often misapply them, or misinterpret them, or do not think about the meaning of the situations in which they might be successfully applied.[1]

Many of our discussions will revolve around specific practical activities that can be used with all grades so that, as a larger group, we can examine trajectories of learning. Activities include:

- exploring function machine, robot rules activities
- building linear growing patterns
- greating graphs
- solving contextual problems

Throughout these activities and tasks we will return consistently to Mason's idea of noticing by listening carefully and fearlessly to one another and responding through our deepening understanding of mathematics.

[1] To solve equations of the form ax+b=cx+d, students are taught a standard procedure using subtraction in order to get the variable terms on the left and the constant on the right, and then dividing by the coefficient of the variable term. However, a more meaningful approach is to compare the different rates of growth as represented by the multipliers, and subtraction to compare the constants, or "where the two lines start on a graph", and finally, divided that number by the rate of growth number. If we think of ax+b and cx+d as two pattern rules for which the value of the multiplier and the constant are different, then the solution to the equation is analogous to finding the position number (x) at which the trend lines of the two rules will intersect. To determine how far apart trend lines "start" on the y-axis, students find the numeric difference between the values of the constant, or (d-b). To find the rate at which they "come together," students find the difference between one multiplier and the other (a-c). To find the position number (x), they divide "how far apart they started" by "the rate at which they come together" or x=(d-b)÷(a-c). This helps to support a more conceptual understanding of *why* the operations of subtraction and division are carried out when solving for x.

Proposed supporting articles:

Ferrini-Mundy, Lappan & Phillips. (2000). Experiences with patterning. In Moses, B (Ed.), *Algebraic Thinking, Grades K-12; Readings from NCTM's School-Based Journals and Other Publications*. Reston, VA: National Council of Teachers of Mathematics, pp. 282-289.

Siemon, D & Breed, M. Assessing multiplicative thinking using rich tasks.

Mason, John (2009). Teaching as disciplined enquiry. *Teachers and Teaching: Theory and practice*, 15(2), pp. 205–223.

Group	Multiplicative Thinking	Generalizing	Multiple Representations
	• difference between additive and multiplicative reasoning • additive tells about the difference between quantities (variation of one set of data) • multiplicative reasoning is about ratio between quantities (co-variation between two sets of data) • developing one-to-many correspondence	• how to emphasize situations in which generalizations can be identified and described	• Utilizing and developing students' natural ability to discern patterns (in multiple forms) make connections among forms, and generalize
K-4	What does multiplicative thinking look like for young learners? E.g., multiplying a number by 2 might be thought of as: doubling a number, adding a number to itself, number plus number, number x 2. Multiplying by 3 might be thought of as: Doubling the number and adding 1 more, Number + number + number, Tripling the number. Build on students' emergent capacity to skip count. Co-variation between sets of data vs. variation within a set of data	How young children can start to think about generalization – using patterns to start to identify and articulate mathematical structure. Constructing pattern rules. Making predictions. Questions that can foster the habit of looking for patterns and relationships.	Building growing patterns and understanding the relationship between the position number and number of tiles. Making predictions based on pattern rules. Connecting this understanding to Function Machine games. Finding rules for the function machine – making predictions for input and output numbers.

	Multiplicative Thinking	Generalizing	Multiple Representations
3-6	How does multiplicative thinking develop? Building linear growing patterns – developing an idea of the unit (the part of the pattern that is repeated as a function of the position number). Swimming Pool Problem - counting tiles – transitioning from additive to multiplicative thinking. For instance when counting tiles for the 2^{nd}, 3^{rd}, 4^{th}, nth pool beginning to understand the structure of the array, adding rows or columns. Understanding the multiplicative relationships involved in mathematical structure (from additive to multiplicative reasoning – recursion to function)	Finding and articulating generalized rules for patterns. Building patterns from generalized rules. Use rules to find missing output numbers (dependent variables) or missing input number (independent variables). Questions that can foster the habit of looking for patterns and relationships.	Exploring the connections between the Function Machine or Robot guess my rule game, linear growing patterns, and graphs. Exploring equivalence of expression – multiple rules for the Swimming Pool Problem. Connections to story problems.

	Multiplicative Thinking	Generalizing	Multiple Representations
5-8	How does multiplicative thinking develop? Building linear growing patterns – developing an idea of the unit (the part of the pattern that is repeated as a function of the position number). Swimming Pool Problem – (Grades 5 and 6) Counting tiles – transitioning from additive to multiplicative thinking. For instance when counting tiles for the 2nd, 3rd, 4th, nth pool beginning to understand the structure of the array, adding rows or columns. Understanding the multiplicative relationships involved in mathematical structure (from additive to multiplicative reasoning – recursion to function)	Finding and articulating generalized rules for patterns. Building patterns from generalized rules. Looking at different kinds of generalizations. Comparing linear and quadratic relationships – recognizing and expressing different relationships. Questions that can foster the habit of looking for patterns and relationships.	Exploring pattern rules, patterns and graphs (some equations). Graphs of linear growing patterns. Graphs of swimming pool – quadratic vs. linear. Move to solving equations based on an understanding of what x and y actually represent – having students construct an understanding for solving equations of the form ax+b=cx+d (what does the point of intersection on the graph represent?) Developing an understanding of x as a generalized number that can take a range of values (bridge from the idea of unknown to that of variable) – so x is any position number of a pattern, and y is the number of tiles dependent on that number, given the pattern rule

	Multiplicative Thinking	Generalizing	Multiple Representations
7-10	How does multiplicative thinking develop? Building linear growing patterns – developing an idea of the unit (the part of the pattern that is repeated as a function of the position number). Swimming Pool Problem – comparing types of multiplicative relationships (linear, quadratic). Understanding the multiplicative relationships involved in mathematical structure (from additive to multiplicative reasoning – recursion to function)	Finding and articulating generalized rules for patterns. Building patterns from generalized rules. Looking at different kinds of generalizations. Comparing linear and quadratic relationships – recognizing and expressing different relationships. Questions that can foster the habit of looking for patterns and relationships. Developing an understanding of x as a generalized number that can take a range of values (bridge from the idea of unknown to that of variable) – so x is any position number of a pattern, and y is the number of tiles dependent on that number, given the pattern rule	Exploring pattern rules, patterns, graphs, equations and connections among representations. Move to solving equations based on an understanding of what x and y actually represent – having students <u>construct</u> an understanding for solving equations of the form ax+b=cx+d.

	Multiplicative Thinking	Generalizing	Multiple Representations
9-12	How does multiplicative thinking develop? Building linear growing patterns – developing an idea of the unit (the part of the pattern that is repeated as a function of the position number). Swimming Pool Problem – comparing types of multiplicative relationships (linear, quadratic). Understanding the multiplicative relationships involved in mathematical structure (from additive to multiplicative reasoning – recursion/scalar to function)	Finding and articulating generalized rules for patterns. Building patterns from generalized rules. Looking at different kinds of generalizations. Comparing linear and quadratic relationships – recognizing and expressing different relationships. Developing an understanding of x as a generalized number that can take a range of values (bridge from the idea of unknown to that of variable) – so x is any position number of a pattern, and y is the number of tiles dependent on that number, given the pattern rule Questions that can foster the habit of looking for patterns and relationships.	Exploring pattern rules, patterns, graphs, equations and connections among representations. Move to solving equations based on an understanding of what x and y actually represent – having students construct an understanding for solving equations of the form ax+b=cx+d.